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Enriched and internal categories: an extensive relationship*

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Abstract

We consider an extant infinitary variant of Lawvere's finitary definition of extensivity of a category $\mathcal{V}$. In the presence of cartesian closedness and finite limits in $\mathcal{V}$, we give two characterisations of the condition in terms of a biequivalence between the bicategory of matrices over $\mathcal{V}$ and the bicategory of spans over discrete objects in $\mathcal{V}$. Using the condition, we prove that $\mathcal{V}$-**Cat** and the category $\mathbf{Cat}_{\mathrm{d}}(\mathcal{V})$ of internal categories in $\mathcal{V}$ with a discrete object of objects are equivalent. Our leading example has $\mathcal{V} = \mathbf{Cat}$, making $\mathcal{V}$-**Cat** the category of all small 2-categories and $\mathbf{Cat}_{\mathrm{d}}(\mathcal{V})$ the category of small double categories with discrete category of objects. We further show that if $\mathcal{V}$ is extensive, then so are $\mathcal{V}$-**Cat** and $\mathbf{Cat}(\mathcal{V})$, allowing the process to iterate.

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1 Introduction

A standard definition of category [13, Section I.8] asks for a set of objects $\mathrm{ob}(\mathcal{C})$ together with, for each A and B in $\mathrm{ob}(\mathcal{C})$, a set $\mathcal{C}(A, B)$, together with data and axioms for composition and identities. Another standard definition of category [13, Section I.2] asks for a set of objects $\mathrm{ob}(\mathcal{C})$, together with a set $\mathrm{ar}(\mathcal{C})$ of arrows, and functions $\mathrm{dom}, \mathrm{cod} \colon \mathrm{ar}(\mathcal{C}) \longrightarrow \mathrm{ob}(\mathcal{C})$, together with composition and identity data and axioms. The two definitions are equivalent: given an instance of the former, one may put $\mathrm{ar}(\mathcal{C}) = \coprod_{A,B \in \mathrm{ob}(\mathcal{C})} \mathcal{C}(A, B)$ to obtain an instance of the latter, and conversely, given an instance of the latter, one may put $\mathcal{C}(A, B) = \{ f \in \mathrm{ar}(\mathcal{C}) \mid \mathrm{dom}(f) = A \text{ and } \mathrm{cod}(f) = B \}$ to obtain an instance of the former, these constructions being mutually inverse up to isomorphism. The first definition extends to that of enriched category [10] while the second extends to that of internal category [9, Section B.2.3]. The relationship between these two generalisations of the notion of category is the topic of this paper.

Our leading example of the relationship is that between 2-categories and double categories: a 2-category is a **Cat**-enriched category, while a double category is a category in **Cat**. There is a natural adjunction between the categories **2-Cat** = **Cat-Cat** of 2-categories and **DblCat** = **Cat**(**Cat**) of double categories. The left adjoint **2-Cat** $\longrightarrow$ **DblCat** maps a 2-category to the corresponding double category whose vertical morphisms are all identities, i.e., whose category of objects is discrete: our convention is that a double category is a category in **Cat** whose category of objects consists of objects and vertical morphisms and whose category of arrows consists of

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horizontal morphisms and 2-cells. This functor is fully faithful, allowing us to identify 2-categories with “vertically discrete” double categories. The right adjoint **DblCat** $\longrightarrow$ **2-Cat** forgets the vertical structure of a double category.

We extend this comparison between enrichment and internalisation from **Cat** to an arbitrary base category $\mathcal{V}$ subject to axiomatically defined conditions. Specifically, we will require $\mathcal{V}$ to be cartesian closed and have finite limits, together with small coproducts satisfying a stengthening from a finitary condition to an infinitary condition of Lawvere's [11, 12] notion of extensivity as developed by Carboni et al. [6]. This infinitary version of extensivity has been studied by Borceux and Janelidze [4], and by Centazzo and Vitale [7], probably also by others, and is an instance of the notion of van Kampen colimit [8].

Cartesianness of $\mathcal{V}$ allows one to define the category $\mathcal{V}$-**Cat**. Finite limits in $\mathcal{V}$ allow one to define the category $\mathbf{Cat}(\mathcal{V})$. Adding closedness and an extensivity condition allows one to construct a functor $\mathrm{Int}\colon \mathcal{V}\text{-}\mathbf{Cat} \longrightarrow \mathbf{Cat}(\mathcal{V})$ with a right adjoint En, and to prove that, if one restricts the image of Int to the category $\mathbf{Cat}_{\mathrm{d}}(\mathcal{V})$ of categories in $\mathcal{V}$ with a “discrete” object of objects, i.e., with object of objects given by a copower of the terminal object, then Int generates an equivalence of categories between $\mathcal{V}$-**Cat** and $\mathbf{Cat}_{\mathrm{d}}(\mathcal{V})$.

Informally, the idea is that a $\mathcal{V}$-category $\mathcal{C}$ has a set of objects $\mathrm{ob}(\mathcal{C})$, which one sends to the $\mathrm{ob}(\mathcal{C})$-copower of the terminal object of $\mathcal{V}$, and one sums the homs of $\mathcal{C}$ to give the object of arrows of $\mathrm{Int}(\mathcal{C})$. One requires small coproducts in order to make the sum, and one requires an infinitary extensivity condition in order to define composition in $\mathrm{Int}(\mathcal{C})$ and to prove the equivalence. Two aspects of this require care.

First, the definition of composition in $\mathrm{Int}(\mathcal{C})$ involves a non-trivial coherence condition, and proving its associativity requires a calculation. Although associativity can be proved directly, it may be seen as a consequence of a characterisation of infinitary extensivity as the condition that a natural oplax functor, denoted by $\mathscr{I}nt$, from the bicategory $\mathcal{V}$-$\mathscr{M}at$ of matrices over $\mathcal{V}$ to the bicategory $\mathscr{S}pan_{\mathrm{d}}(\mathcal{V})$ of spans in $\mathcal{V}$ over discrete objects, is a biequivalence of bicategories. Infinitary extensivitiy may also be characterised by the assertion that a natural lax functor $\mathscr{E}n$ from $\mathscr{S}pan_{\mathrm{d}}(\mathcal{V})$ to $\mathcal{V}$-$\mathscr{M}at$ is a biequivalence of bicategories. One does not need these characterisations in order to establish the equivalence between $\mathcal{V}$-**Cat** and $\mathbf{Cat}_{\mathrm{d}}(\mathcal{V})$, but they provide a natural setting for it.

Second, in general, $\mathrm{Int}\colon \mathcal{V}\text{-}\mathbf{Cat} \longrightarrow \mathbf{Cat}(\mathcal{V})$ is not fully faithful. This failure occurs even in the case where $\mathcal{V} = \mathbf{Set} \times \mathbf{Set}$. To see this, observe that a $(\mathbf{Set} \times \mathbf{Set})$-category consists of a pair of categories $\mathcal{C}$ and $\mathcal{C}'$ such that $\mathrm{ob}(\mathcal{C}) = \mathrm{ob}(\mathcal{C}')$, and a $(\mathbf{Set} \times \mathbf{Set})$-functor from $(\mathcal{C}, \mathcal{C}')$ to $(\mathcal{D}, \mathcal{D}')$ is a pair of functors $F\colon \mathcal{C} \longrightarrow \mathcal{D}$ and $F'\colon \mathcal{C}' \longrightarrow \mathcal{D}'$ whose behaviour agrees on objects. On the other hand, a category in $\mathbf{Set} \times \mathbf{Set}$ is an arbitrary pair of categories, and an internal functor is an arbitrary pair of functors. Hence the functor Int, which in this case is the obvious inclusion, is not full. The functor Int also need not be faithful: putting $\mathcal{V} = 1$, $\mathcal{V}$-**Cat** is equivalent to **Set**, whereas $\mathbf{Cat}(\mathcal{V})$ is equivalent to 1. So, in general, the replacement of $\mathbf{Cat}(\mathcal{V})$ by $\mathbf{Cat}_{\mathrm{d}}(\mathcal{V})$ is not trivial.

The paper is organised as follows. In Section 2, we follow Borceux and Janelidze [4] and Centazzo and Vitale [7] in extending Carboni et al.'s development [6] of Lawvere's notion of extensivity from a finitary condition to an infinitary condition, and in extending generic results about it. We also prove that, if $\mathcal{V}$ is extensive, so are $\mathcal{V}$-**Cat** and $\mathbf{Cat}(\mathcal{V})$. In Section 3, we introduce $\mathcal{V}$-matrices and spans in $\mathcal{V}$ and use them to characterise infinitary extensivity in the presence of cartesian closed structure and finite limits. In Section 4, we state and prove our main results relating enriched categories with internal categories.

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2 Extensivity

Let $\mathcal{V}$ be a category, I a set, and $(X_i)_{i\in I}$ an I-indexed family of objects of $\mathcal{V}$. If $\mathcal{V}$ admits all I-indexed coproducts, there is a functor

$$\prod_{i\in I}(\mathcal{V}/X_i) \xrightarrow{\quad\coprod\quad} \mathcal{V}/(\textstyle\coprod_{i\in I} X_i) \tag{1}$$

mapping a family $(f_i\colon A_i \longrightarrow X_i)_{i\in I}$ to $\coprod_{i\in I} f_i\colon \coprod_{i\in I} A_i \longrightarrow \coprod_{i\in I} X_i$.

We follow Borceux and Janelidze [4, Chap 2, 6.3] and Centazzo and Vitale [7, 4.1] in making the following definition.

Definition 2.1. A category $\mathcal{V}$ is *extensive* if $\mathcal{V}$ admits all small coproducts and, for each small family $(X_i)_{i\in I}$ of objects of $\mathcal{V}$, the functor $\coprod$ in (1) is an equivalence of categories.

This definition of extensivity is a strengthening of that of [6], which may be obtained by replacing "small" by "finite". The generic results of [6] extend routinely. Consider the functor $\coprod$ in (1), initially assuming only that $\mathcal{V}$ has all I-indexed coproducts.

Proposition 2.2. If $\mathcal{V}$ has all I-indexed coproducts, the functor $\coprod$ in (1) admits a right adjoint if and only if for every $i \in I$, the category $\mathcal{V}$ has all pullbacks along the i-th coprojection $\sigma_i\colon X_i \longrightarrow \coprod_{i\in I} X_i$.

The right adjoint of $\coprod$, if it exists, has i-th component $\sigma_i^*\colon \mathcal{V}/(\coprod_{i\in I} X_i) \longrightarrow \mathcal{V}/X_i$ mapping $f\colon A \longrightarrow \coprod_{i\in I} X_i$ to $\sigma_i^* f\colon \sigma_i^* A \longrightarrow X_i$ defined by the pullback

$$\begin{array}{ccc} \sigma_i^* A & \longrightarrow & A \\ {\scriptstyle \sigma_i^* f}\downarrow & \lrcorner & \downarrow{\scriptstyle f} \\ X_i & \xrightarrow{\ \sigma_i\ } & \coprod_{i\in I} X_i \end{array}$$

in $\mathcal{V}$. Thus, in the presence of such pullbacks, we have the adjunction

$$\prod_{i\in I}(\mathcal{V}/X_i) \underset{\langle \sigma_i^* \rangle_{i\in I}}{\overset{\coprod}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \mathcal{V}/(\textstyle\coprod_{i\in I} X_i)\ .$$

Proposition 2.3. If $\mathcal{V}$ has all I-indexed coproducts, the functor $\coprod$ in (1) is an equivalence of categories if and only if $\mathcal{V}$ has all pullbacks along the coprojections σ_i and, given a morphism $f\colon A \longrightarrow \coprod_{i\in I} X_i$ and commutative squares

$$\begin{array}{ccc} A_i & \xrightarrow{\ \tau_i\ } & A \\ {\scriptstyle f_i}\downarrow & & \downarrow{\scriptstyle f} \\ X_i & \xrightarrow{\ \sigma_i\ } & \coprod_{i\in I} X_i \end{array} \tag{2}$$

for all $i \in I$, each square (2) is a pullback exactly when the morphisms $\tau_i\colon A_i \longrightarrow A$ define a coproduct in $\mathcal{V}$.

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Proof. By Proposition 2.2, $\coprod$ has a right adjoint if and only if $\mathcal{V}$ has pullbacks along each σ_i. The final clause of the proposition asserts that the adjunction is an equivalence if and only if its unit and counit are isomorphisms. Q.E.D.

Corollary 2.4 ([7, Section 4.2 Exercise 1],Cf. [6, Proposition 2.2])**.** A category $\mathcal{V}$ with small coproducts is extensive if and only if it has all pullbacks along coprojections associated with small coproducts, and for every small family $(X_i)_{i\in I}$ of objects, a morphism $f\colon A \longrightarrow \coprod_{i\in I} X_i$ and commutative squares (2), each square (2) is a pullback exactly when the morphisms $\tau_i\colon A_i \longrightarrow A$ define a coproduct in $\mathcal{V}$.

For a set I and an object X of $\mathcal{V}$, we denote the I-fold copower of X by $I \bullet X$. If $\mathcal{V}$ has a terminal object 1, then as a special case of (1) where $X_i = 1$ for all $i \in I$, we obtain the functor

$$\mathcal{V}^I \xrightarrow{\quad \coprod \quad} \mathcal{V}/(I \bullet 1)\ , \tag{3}$$

using the isomorphism $\mathcal{V}/1 \cong \mathcal{V}$.

By Proposition 2.2, if $\mathcal{V}$ admits all pullbacks along the coprojections $\ulcorner i \urcorner\colon 1 \longrightarrow I \bullet 1$ for all $i \in I$, there is an adjunction

$$\mathcal{V}^I \underset{\langle \ulcorner i \urcorner^* \rangle_{i\in I}}{\overset{\coprod}{\underset{\perp}{\rightleftarrows}}} \mathcal{V}/(I \bullet 1)\ . \tag{4}$$

Proposition 2.5 (Cf. [6, Proposition 4.1])**.** Let $\mathcal{V}$ be a category with small coproducts and a terminal object. If for every small set I, the functor $\coprod$ in (3) is an equivalence, then $\mathcal{V}$ is extensive.

Proof. The proof follows that of [6, Proposition 4.1]. We shall use Corollary 2.4; given a small family $(X_i)_{i\in I}$ of objects, a morphism $f\colon A \longrightarrow \coprod_{i\in I} X_i$ and commutative squares (2), consider the following commutative diagrams for all $i \in I$.

$$\begin{array}{ccc}
A_i & \xrightarrow{\ \tau_i\ } & A \\
{\scriptstyle f_i}\downarrow & & \downarrow{\scriptstyle f} \\
X_i & \xrightarrow{\ \sigma_i\ } & \coprod_{i\in I} X_i \\
{\scriptstyle !}\downarrow & & \downarrow{\scriptstyle \coprod_{i\in I} !} \\
1 & \xrightarrow{\ \ulcorner i \urcorner\ } & I \bullet 1
\end{array}$$

As (3) is an equivalence, by Proposition 2.3 the lower squares are pullbacks. Thus, the upper squares are pullbacks if and only if the whole rectangles are pullbacks, which in turn is true if and only if the τ_i's define a coproduct. Q.E.D.

Example 2.1. **Set** is extensive. So, for any small category $\mathcal{C}$, the presheaf category $[\mathcal{C}, \mathbf{Set}]$ is also extensive by Corollary 2.4, as pullbacks and small coproducts are given pointwise. More generally, if $\mathcal{V}$ is extensive, then so is the functor category $[\mathcal{C}, \mathcal{V}]$.

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Example 2.2 ([4, Chap 2, Section 6.3.2], Cf. [6, Proposition 2.4])**.** For any locally small category $\mathcal{V}$, the free small coproduct completion $\mathbf{Fam}(\mathcal{V})$ is extensive. To see this, let $((X_{i,j})_{j\in J_i})_{i\in I}$ be an I-indexed family of objects in $\mathbf{Fam}(\mathcal{V})$. An object of the category $\mathbf{Fam}(\mathcal{V})/(\coprod_{i\in I}(X_{i,j})_{j\in J_i})$,

$$(A_k)_{k\in K} \longrightarrow \coprod_{i\in I}(X_{i,j})_{j\in J_i} = (X_{i,j})_{(i,j)\in\coprod_{i\in I} J_i} \ ,$$

consists of a function $f\colon K \longrightarrow \coprod_{i\in I} J_i$ and, for each $k \in K$, a morphism $A_k \longrightarrow X_{f(k)}$ in $\mathcal{V}$.

Write $f^{-1}(i) = \{\, k \in K \mid f(k) = (i,j) \text{ for some } j \in J_i \,\}$. Then the object

$$((A_k)_{k\in f^{-1}(i)} \longrightarrow (X_{i,j})_{j\in J_i})_{i\in I}$$

of $\prod_{i\in I}(\mathbf{Fam}(\mathcal{V})/(X_{i,j})_{j\in J_i})$ maps to the object of $\mathbf{Fam}(\mathcal{V})/(\coprod_{i\in I}(X_{i,j})_{j\in J_i})$ above, so $\coprod$ is essentially surjective. Fully faithfulness follows from the fact that $\coprod$ essentially just takes disjoint unions over indexing sets.

Proposition 2.6. If $\mathcal{V}$ is a monoidal category with a strict initial object 0 for which $X \otimes 0 \cong 0 \otimes X \cong 0$ for every $X \in \mathcal{V}$, then $\mathcal{V}$-$\mathbf{Cat}$ is extensive.

Proof. The coproduct $\coprod_{i\in I}\mathcal{C}_i$ in $\mathcal{V}$-$\mathbf{Cat}$ of $(\mathcal{C}_i)_{i\in I}$ may be described as follows: $\mathrm{ob}(\coprod_{i\in I}\mathcal{C}_i) = \coprod_{i\in I}\mathrm{ob}(\mathcal{C}_i)$ and

$$(\coprod_{i\in I}\mathcal{C}_i)((i,C),(i',C')) = \begin{cases}\mathcal{C}_i(C,C') & \text{if } i=i'\\ 0 & \text{otherwise.}\end{cases}$$

Given a functor $H\colon \mathcal{A} \longrightarrow \coprod_{i\in I}\mathcal{C}_i$, for each $i \in I$ define $\mathcal{A}_i$ to be the full subcategory of $\mathcal{A}$ consisting of those objects A for which $HA = (i,C)$ for some $C \in \mathcal{C}_i$. Then $\mathcal{A} \cong \coprod_{i\in I}\mathcal{A}_i$, because for objects $A \in \mathcal{A}_i \subseteq \mathcal{A}$ and $A' \in \mathcal{A}_{i'} \subseteq \mathcal{A}$ with $i \neq i'$, we have $\mathcal{A}(A,A') \cong 0$ since there is a morphism $H_{A,A'}\colon \mathcal{A}(A,A') \longrightarrow (\coprod_{i\in I}\mathcal{C}_i)(HA,HA') = 0$ in $\mathcal{V}$. Q.E.D.

Putting $I = \varnothing$, Proposition 2.3 asserts that if $\mathcal{V}$ has an initial object 0, the functor $\coprod\colon 1 \longrightarrow \mathcal{V}/0$ is an equivalence of categories if and only if given any morphism $f\colon A \longrightarrow 0$, f is necessarily an isomorphism. So if a category is extensive, it has a strict initial object. This allows to deduce the following.

Corollary 2.7. If $\mathcal{V}$ is a symmetric monoidal closed extensive category with small limits, then so is $\mathcal{V}$-$\mathbf{Cat}$.

Applying Corollary 2.7 inductively, starting with Example 2.1, we may conclude that the category $\mathbf{n}$-$\mathbf{Cat}$ of strict n-categories for each natural number n is extensive, thus in particular, $\mathbf{Cat}$ and $\mathbf{2}$-$\mathbf{Cat}$ are extensive.

Next we study when the category $\mathbf{Cat}(\mathcal{V})$ of categories internal to a category $\mathcal{V}$ with finite limits is extensive. An internal category $\mathcal{C}$ consists of objects C_0, C_1 of $\mathcal{V}$, morphisms $\mathrm{dom}, \mathrm{cod}\colon C_1 \longrightarrow C_0$, $e\colon C_0 \longrightarrow C_1$ and $c\colon C_2 \longrightarrow C_1$, where C_2 is the pullback in $\mathcal{V}$ of dom and cod.

Proposition 2.8. Let $\mathcal{V}$ be a category with finite limits and small coproducts. If, for every small family of objects $(X_i)_{i\in I}$ of $\mathcal{V}$, the functor $\coprod$ in (1) preserves binary products, then $\mathbf{Cat}(\mathcal{V})$ has small coproducts.

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Proof. Given a family $(\mathcal{C}^{(i)})_{i\in I}$ of objects of $\mathbf{Cat}(\mathcal{V})$, put $X_i = C_0^{(i)}$ for each i. Then for each i, $C_2^{(i)}$ is (the domain of) the binary product in $\mathcal{V}/C_0^{(i)}$ of $\mathrm{dom}^{(i)}: C_1^{(i)} \longrightarrow C_0^{(i)}$ and $\mathrm{cod}^{(i)}: C_1^{(i)} \longrightarrow C_0^{(i)}$. Thus $(C_2^{(i)} \longrightarrow C_0^{(i)})_{i\in I}$ is the binary product in $\prod_{i\in I}(\mathcal{V}/C_0^{(i)})$ of $(\mathrm{dom}^{(i)})_{i\in I}$ and $(\mathrm{cod}^{(i)})_{i\in I}$.

Since $\coprod$ preserves binary products, we see that $\coprod_{i\in I} C_2^{(i)}$ is the pullback of $\coprod_{i\in I}\mathrm{dom}^{(i)}: \coprod_{i\in I} C_1^{(i)} \longrightarrow \coprod_{i\in I} C_0^{(i)}$ and $\coprod_{i\in I}\mathrm{cod}^{(i)}: \coprod_{i\in I} C_1^{(i)} \longrightarrow \coprod_{i\in I} C_0^{(i)}$. This ensures that we can define $\coprod_{i\in I}\mathcal{C}^{(i)}$ simply by $(\coprod_{i\in I}\mathcal{C}^{(i)})_0 = \coprod_{i\in I} C_0^{(i)}$ and $(\coprod_{i\in I}\mathcal{C}^{(i)})_1 = \coprod_{i\in I} C_1^{(i)}$, with relevant morphisms induced from each $\mathcal{C}^{(i)}$ by taking coproduct. The remainder of the proof is routine. Q.E.D.

Theorem 2.9. If $\mathcal{V}$ is an extensive category with finite limits, then so is $\mathbf{Cat}(\mathcal{V})$.

Proof. Let $\mathbb{G}$ be the category with two objects, 0 and 1, and two non-trivial morphisms $s, t: 1 \longrightarrow 0$, and write $\mathbf{Graph}(\mathcal{V}) = [\mathbb{G}, \mathcal{V}]$. Then there is a functor $U: \mathbf{Cat}(\mathcal{V}) \longrightarrow \mathbf{Graph}(\mathcal{V})$ which sends a category in $\mathcal{V}$ to its underlying graph in $\mathcal{V}$; U reflects isomorphisms and preserves pullbacks. Since $\mathcal{V}$ is extensive, $\coprod$ preserves binary products, so by Proposition 2.8, $\mathbf{Cat}(\mathcal{V})$ has all small coproducts, and U preserves them.

So in particular, U preserves and reflects pullbacks and small coproducts. By Corollary 2.4, extensivity can be characterised in terms of pullbacks and small coproducts. Thus, since $\mathbf{Graph}(\mathcal{V})$ is extensive (being a functor category; see Example 2.1), $\mathbf{Cat}(\mathcal{V})$ is also extensive. Q.E.D.

For our final class of examples, we call a small family $(G_i)_{i\in I}$ of objects of a category $\mathcal{V}$ with coproducts a *connected strong generator* if (i) each G_i is *connected* in the sense that $\mathcal{V}(G_i, -): \mathcal{V} \longrightarrow \mathbf{Set}$ preserves small coproducts, and (ii) denoting by $\mathcal{G}$ the full subcategory of $\mathcal{V}$ determined by the objects $(G_i)_{i\in I}$, the functor $U: \mathcal{V} \longrightarrow [\mathcal{G}^{\mathrm{op}}, \mathbf{Set}]$ mapping X to the functor $(G_i \mapsto \mathcal{V}(G_i, X))$ reflects isomorphisms.

Proposition 2.10. Let $\mathcal{V}$ have finite limits, small coproducts and a connected strong generator. Then $\mathcal{V}$ is extensive.

Proof. U preserves finite limits and small coproducts. So, as U reflects isomorphisms, it follows that U reflects finite limits and small coproducts. As $[\mathcal{G}^{op}, \mathbf{Set}]$ is extensive by Example 2.1, Corollary 2.4 yields the result. Q.E.D.

The conditions on $\mathcal{V}$ in Proposition 2.10 are similar to conditions used by Verity [15, Section 2.1] in a comparison between enriched categories and internal categories.

3 $\mathcal{V}$-matrices and spans in $\mathcal{V}$

Before comparing enriched categories with internal categories in Section 4, here we characterise extensivity in terms of the relationship between bicategories of $\mathcal{V}$-matrices and spans in $\mathcal{V}$, cf. [14]. Readers unfamiliar with bicategories can safely skip this section.

We assume throughout that $\mathcal{V}$ is cartesian closed and has all finite limits and small coproducts.

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3.1 $\mathcal{V}$-matrices

The bicategory $\mathcal{V}$-$\mathscr{M}at$ of *$\mathcal{V}$-matrices* [2] has small sets as objects, and hom-categories given by $\mathcal{V}\text{-}\mathscr{M}at(I,J) = \mathcal{V}^{I\times J}$. Explicitly, a 1-cell $M\colon I \longrightarrow J$ is a family $(M(i,j))_{i\in I, j\in J}$ of objects of $\mathcal{V}$, i.e., an $(I \times J)$-matrix whose components are objects of $\mathcal{V}$. The identity 1-cell Id_I on a set I is given by

$$\mathrm{Id}_I(i,i') = \begin{cases} 1 & \text{if } i = i' \\ 0 & \text{otherwise,} \end{cases}$$

where by 1 and 0 we mean the terminal and initial objects of $\mathcal{V}$ respectively. Composition of 1-cells is defined by matrix multiplication: given $M = (M(i,j))_{i\in I,j\in J}$ and $N = (N(j,k))_{j\in J,k\in K}$, their composite $N \circ M$ is defined by

$$(N \circ M)(i,k) = \coprod_{j\in J} N(j,k) \times M(i,j)\ .$$

A 2-cell from M to $M' = (M'(i,j))_{i\in I,j\in J}$ consists of a family $(f(i,j)\colon M(i,j) \longrightarrow M'(i,j))_{i\in I,j\in J}$ of morphisms of $\mathcal{V}$, with the composition of 1-cells given above readily extending to define horizontal composition of 2-cells.

The closed structure of $\mathcal{V}$ yields canonical unit and associativity isomorphisms. For example, given a matrix $L = (L(h,i))_{h\in H,i\in I}$ together with M and N, we have

$$\begin{aligned} ((N \circ M) \circ L)(h,k) &= \coprod_{i\in I}\left(\coprod_{j\in J} N(j,k) \times M(i,j)\right) \times L(h,i) \\ &\cong \coprod_{i\in I, j\in J} N(j,k) \times M(i,j) \times L(h,i) \\ &\cong \coprod_{j\in J} N(j,k) \times \left(\coprod_{i\in I} M(i,j) \times L(h,i)\right) \\ &= (N \circ (M \circ L))(h,k)\ , \end{aligned}$$

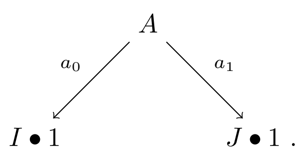
using the fact that for each object X of $\mathcal{V}$, the functor $- \times X$ preserves small coproducts.

3.2 Spans in $\mathcal{V}$

The bicategory $\mathscr{S}pan(\mathcal{V})$ of spans in $\mathcal{V}$ was introduced by Bénabou [1]. We need a variant that we denote by $\mathscr{S}pan_{\mathrm{d}}(\mathcal{V})$.

As $\mathcal{V}$ has small coproducts, the representable functor $\mathcal{V}(1,-)\colon \mathcal{V} \longrightarrow \mathbf{Set}$ has a left adjoint given by $(-) \bullet 1$. It is routine to verify that $(-) \bullet 1$ preserves finite products.

The bicategory $\mathscr{S}pan_{\mathrm{d}}(\mathcal{V})$ has small sets as objects, with hom-categories given by $\mathscr{S}pan_{\mathrm{d}}(\mathcal{V})(I,J) = \mathcal{V}/((I \bullet 1) \times (J \bullet 1)) \cong \mathcal{V}/((I \times J) \bullet 1)$. We depict a 1-cell $a = \langle a_0, a_1\rangle\colon A \longrightarrow (I \bullet 1) \times (J \bullet 1)$ from I to J as follows:



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Composition of 1-cells

$$\begin{array}{ccc} & A & \\ {\scriptstyle a_0}\swarrow & & \searrow{\scriptstyle a_1} \\ I \bullet 1 & & J \bullet 1 \end{array} \quad \text{and} \quad \begin{array}{ccc} & B & \\ {\scriptstyle b_0}\swarrow & & \searrow{\scriptstyle b_1} \\ J \bullet 1 & & K \bullet 1 \end{array}$$

is given by the pullback

$$\begin{array}{ccccc} & & A \times_{J\bullet 1} B & & \\ & {\scriptstyle \pi_0}\swarrow & \lrcorner & \searrow{\scriptstyle \pi_1} & \\ & A & & B & \\ {\scriptstyle a_0}\swarrow & & {\scriptstyle a_1}\searrow \quad \swarrow{\scriptstyle b_0} & & \searrow{\scriptstyle b_1} \\ I \bullet 1 & & J \bullet 1 & & K \bullet 1 \end{array}$$

in $\mathcal{V}$. A 2-cell is a morphism f in $\mathcal{V}$ making the diagram

$$\begin{array}{ccccc} & & A & & \\ & {\scriptstyle a_0}\swarrow & \downarrow{\scriptstyle f} & \searrow{\scriptstyle a_1} & \\ & & A' & & \\ I \bullet 1 & \xleftarrow{a'_0} & & \xrightarrow{a'_1} & J \bullet 1 \end{array}$$

commute. Composition of 1-cells routinely extends to horizontal composition of 2-cells.

Bénabou's bicategory $\mathscr{S}pan(\mathcal{V})$ has objects of $\mathcal{V}$ as objects, whereas $\mathscr{S}pan_{\mathrm{d}}(\mathcal{V})$ has sets as objects, identifying a set I with the free or "discrete" object of $\mathcal{V}$ on I. There is a fully faithful pseudofunctor $\mathscr{S}pan_{\mathrm{d}}(\mathcal{V}) \longrightarrow \mathscr{S}pan(\mathcal{V})$ mapping I to $I \bullet 1$.

3.3 Relating $\mathcal{V}$-$\mathscr{M}at$ and $\mathscr{S}pan_{\mathrm{d}}(\mathcal{V})$

We now analyse the diagram

$$\mathcal{V}\text{-}\mathscr{M}at \underset{\mathscr{E}n}{\overset{\mathscr{I}nt}{\rightleftarrows}} \mathscr{S}pan_{\mathrm{d}}(\mathcal{V}) \ . \tag{5}$$

Here, $\mathscr{I}nt$ is an oplax functor and $\mathscr{E}n$ a lax functor. Both are the identity on objects and, *locally*, i.e., on the level of actions on hom-categories, they form an adjunction, with $\mathscr{I}nt$ to the left and $\mathscr{E}n$ to the right, cf., the notion of local adjunction [3] and also [5, Theorem 2.39].

3.3.1 The oplax functor $\mathscr{I}nt$

The oplax functor $\mathscr{I}nt$ is defined as follows. On objects, it acts as the identity. The action on hom-categories is given by

$$\mathcal{V}\text{-}\mathscr{M}at(I,J) = \mathcal{V}^{I\times J} \xrightarrow{\quad \coprod \quad} \mathcal{V}/((I \times J)\bullet 1) \cong \mathscr{S}pan_{\mathrm{d}}(\mathcal{V})(I,J)$$

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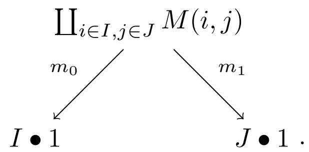
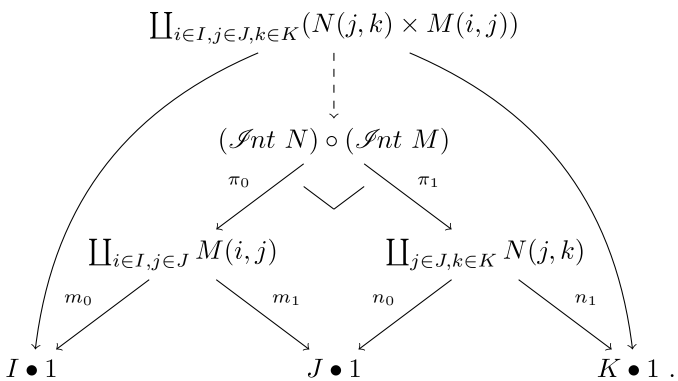
using the left adjoint in (4). So $\mathscr{I}nt$ maps a matrix $(M(i,j))_{i\in I, j\in J}$ to the span



It is routine to verify that $\mathscr{I}nt$ extends canonically to an oplax functor that preserves identities up to coherent isomorphism. The comparison map for composition may be depicted as follows:



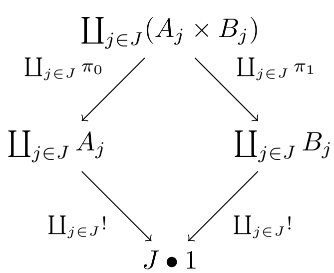
(6)

Proposition 3.1. The oplax functor $\mathscr{I}nt$ is a pseudofunctor if and only if for every small set J, the functor $\coprod\colon \mathcal{V}^J \longrightarrow \mathcal{V}/(J\bullet 1)$ preserves binary products.

Proof. Binary products in $\mathcal{V}^J$ are given pointwise, while binary products in $\mathcal{V}/(J\bullet 1)$ are given by pullbacks. So, if $\coprod\colon \mathcal{V}^J \longrightarrow \mathcal{V}/(J\bullet 1)$ preserves binary products, it sends a family of J binary product diagrams in $\mathcal{V}$, seen as pullbacks over 1, to a pullback over $J\bullet 1$.

By cartesian closedness of $\mathcal{V}$, except for $I\bullet 1$ and $K\bullet 1$ and the maps into them, the diagram (6) is given by the coproduct of a J-indexed family of binary product diagrams seen as pullbacks over 1, together with the comparison map into the pullback. Thus the comparison map is an isomorphism, making $\mathscr{I}nt$ a pseudofunctor.

Conversely, now suppose that $\mathscr{I}nt$ is a pseudofunctor, so the comparison map in (6) is an isomorphism. Let $(A_j)_{j\in J}$, $(B_j)_{j\in J}$ be objects in $\mathcal{V}^J$. Since binary products in $\mathcal{V}/(J\bullet 1)$ are pullbacks, $\coprod$ preserves binary products if the diagram



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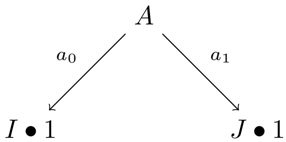
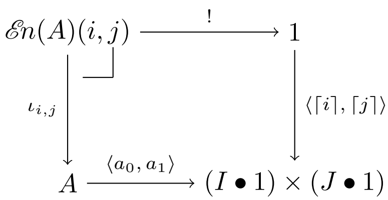
is a pullback square. Consider the matrices $M\colon \{*\} \longrightarrow J$, $N\colon J \longrightarrow \{*\}$ given by $M(*, j) = A_j$, $N(j, *) = B_j$. Applying $\mathscr{I}nt$ to $N \circ M$ gives the square above as an instance of the square in (6); thus it is a pullback square, as required. Q.E.D.

3.3.2 The lax functor $\mathscr{E}n$

The lax functor $\mathscr{E}n\colon \mathscr{S}pan_{\mathrm{d}}(\mathcal{V}) \longrightarrow \mathcal{V}\text{-}\mathscr{M}at$ is the identity on objects, and its action on hom-categories is determined by taking pullbacks. It maps a span



to the matrix $\mathscr{E}n(A) = (\mathscr{E}n(A)(i,j))_{i\in I, j\in J}$ whose (i,j)-th component is given by the pullback



for each $i \in I, j \in J$, where $\lceil i \rceil$ and $\lceil j \rceil$ are the i-th and j-th coprojections respectively. It is routine, using the defining property of a pullback, to verify that $\mathscr{E}n$ is a lax functor.

3.4 A characterisation of extensivity

We now use the constructions of $\mathscr{I}nt$ and $\mathscr{E}n$ to give two characterisations of the notion of extensivity, in the presence of cartesian closedness, finite limits and all small coproducts.

Proposition 3.2. Let $\mathcal{V}$ be a cartesian closed category with finite limits and small coproducts. The oplax functor $\mathscr{I}nt\colon \mathcal{V}\text{-}\mathscr{M}at \longrightarrow \mathscr{S}pan_{\mathrm{d}}(\mathcal{V})$ is a biequivalence if and only if for every set I the adjunction (4) is an adjoint equivalence.

Proof. $\mathscr{I}nt$ is the identity on objects. So we need only consider its local behaviour.

Suppose that, for each set I, the adjunction (4) is an adjoint equivalence. Then, by Proposition 3.1, $\mathscr{I}nt$ is a pseudofunctor. Replacing I by $I \times J$ in the diagram (4) and using the notation for $\mathscr{I}nt$ above, the assertion that (4) is an equivalence makes $\mathscr{I}nt$ locally an equivalence.

Conversely, if $\mathscr{I}nt$ is a biequivalence, it is locally an equivalence. So, putting $J = 1$ yields the result. Q.E.D.

Theorem 3.3. Let $\mathcal{V}$ be a cartesian closed category with finite limits and small coproducts. The following are equivalent.

1. $\mathcal{V}$ is extensive.
2. The oplax functor $\mathscr{I}nt\colon \mathcal{V}\text{-}\mathscr{M}at \longrightarrow \mathscr{S}pan_{\mathrm{d}}(\mathcal{V})$ is a biequivalence.
3. The lax functor $\mathscr{E}n\colon \mathscr{S}pan_{\mathrm{d}}(\mathcal{V}) \longrightarrow \mathcal{V}\text{-}\mathscr{M}at$ is a biequivalence.

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Proof. The equivalence of 1 and 2 is given by Propositions 2.5 and 3.2. That between 2 and 3 follows from the fact of pullback being adjoint to coproduct. Q.E.D.

A monad in the bicategory $\mathcal{V}$-$\mathscr{M}at$ is precisely a small $\mathcal{V}$-category; a monad in the bicategory $\mathscr{S}pan(\mathcal{V})$ is precisely a category in $\mathcal{V}$; and a monad in the bicategory $\mathscr{S}pan_{\mathrm{d}}(\mathcal{V})$ is precisely a category in $\mathcal{V}$ with a "discrete" object of objects. We would therefore like to use Theorem 3.3 immediately to deduce Theorem 4.4 in the next section. However, the situation is more complex than that as maps of monads in $\mathcal{V}$-$\mathscr{M}at$ are not $\mathcal{V}$-functors; similarly for maps of monads in $\mathscr{S}pan(\mathcal{V})$. So we would require additional two-dimensional structure, such as that of a pseudo-double category, in order to deduce Theorem 4.4 from our analysis in this section. In this paper, we give a more elementary account in order to spare those readers without an expert knowledge of two-dimensional category theory.

4 Enrichment and internalization

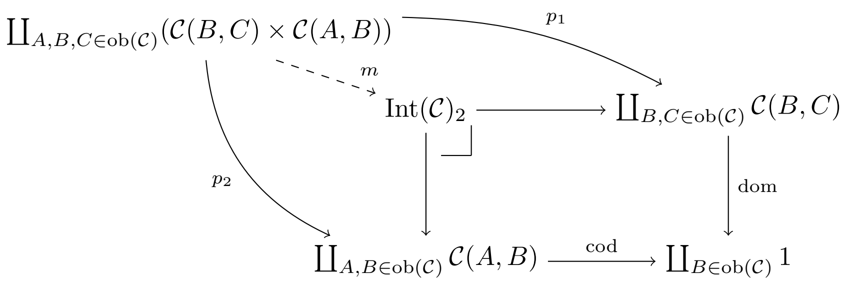
We now establish an adjunction $\mathrm{Int} \dashv \mathrm{En}\colon \mathbf{Cat}(\mathcal{V}) \longrightarrow \mathcal{V}\text{-}\mathbf{Cat}$ and an equivalence of categories between $\mathbf{Cat}_{\mathrm{d}}(\mathcal{V})$, which is the category of categories in $\mathcal{V}$ with a "discrete" object of objects, and $\mathcal{V}$-**Cat**. We assume throughout that $\mathcal{V}$ is cartesian closed and has all finite limits and small coproducts. We will later add extensivity to that.

4.1 The functor $\mathrm{Int}\colon \mathcal{V}\text{-}\mathbf{Cat} \longrightarrow \mathbf{Cat}(\mathcal{V})$

Let $\mathcal{C}$ be a small $\mathcal{V}$-category. We can start to define the category $\mathrm{Int}(\mathcal{C})$ in $\mathcal{V}$ as follows.

- The object of objects is $\mathrm{Int}(\mathcal{C})_0 = \coprod_{A\in \mathrm{ob}(\mathcal{C})} 1 \cong \mathrm{ob}(\mathcal{C}) \bullet 1 \in \mathcal{V}$.
- The object of morphisms is $\mathrm{Int}(\mathcal{C})_1 = \coprod_{A,B\in \mathrm{ob}(\mathcal{C})} \mathcal{C}(A,B) \in \mathcal{V}$, with evident maps dom, cod and e, denoting domain, codomain and identity.

Then we run into difficulty in defining composition, which must be a map of the form $c\colon \mathrm{Int}(\mathcal{C})_2 \longrightarrow \mathrm{Int}(\mathcal{C})_1$. In order to define it, we need the comparison map m in the diagram



(7)

to be invertible. In order to make m invertible, we mimic Proposition 3.1 as follows.

Lemma 4.1. The map m in diagram (7) is invertible if for every set I, the functor $\coprod\colon \mathcal{V}^I \longrightarrow \mathcal{V}/(I\bullet 1)$ preserves binary products.

Proof. Binary products in $\mathcal{V}^I$ are given pointwise, while binary products in $\mathcal{V}/(I\bullet 1)$ are given by pullbacks. So, if $\coprod\colon \mathcal{V}^I \longrightarrow \mathcal{V}/(I\bullet 1)$ preserves binary products, it sends a family of I binary product diagrams in $\mathcal{V}$, seen as pullbacks over 1, to a pullback over $I\bullet 1$.

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By cartesian closedness of $\mathcal{V}$, putting $I = \mathrm{ob}(\mathcal{C})$, diagram (7) is given by the coproduct of an I-indexed family of product diagrams seen as pullbacks over 1, together with the comparison map m into the pullback. So m is an isomorphism. Q.E.D.

By Lemma 4.1, if $\coprod\colon \mathcal{V}^I \longrightarrow \mathcal{V}/(I \bullet 1)$ preserves binary products, we can identify $\mathrm{Int}(\mathcal{C})_2$ with $\coprod_{A,B,C\in\mathrm{ob}(\mathcal{C})}(\mathcal{C}(B,C)\times\mathcal{C}(A,B))$, upon which composition in $\mathrm{Int}(\mathcal{C})$ is defined as follows:

$$\begin{array}{ccc} \mathcal{C}(B,C)\times\mathcal{C}(A,B) & \xrightarrow{\;M_{A,B,C}\;} & \mathcal{C}(A,C) \\ {\scriptstyle\sigma_{A,B,C}}\big\downarrow & & \big\downarrow{\scriptstyle\sigma_{A,C}} \\ \coprod_{A,B,C\in\mathrm{ob}(\mathcal{C})}(\mathcal{C}(B,C)\times\mathcal{C}(A,B)) & \xrightarrow{\;c\;} & \coprod_{A,C\in\mathrm{ob}(\mathcal{C})}\mathcal{C}(A,C) \end{array}$$

It is routine to check that this defines an internal category $\mathrm{Int}(\mathcal{C})$ and that Int extends canonically to a functor $\mathrm{Int}\colon \mathcal{V}\text{-}\mathbf{Cat} \longrightarrow \mathbf{Cat}(\mathcal{V})$.

4.2 The functor $\mathrm{En}\colon \mathbf{Cat}(\mathcal{V}) \longrightarrow \mathcal{V}\text{-}\mathbf{Cat}$

Let $\mathcal{D}$ be a category in $\mathcal{V}$. It consists of objects D_0, D_1 of $\mathcal{V}$, morphisms $\mathrm{dom}, \mathrm{cod}\colon D_1 \longrightarrow D_0$, $e\colon D_0 \longrightarrow D_1$ and $c\colon D_2 \longrightarrow D_1$, where D_2 is a pullback in $\mathcal{V}$. The $\mathcal{V}$-category $\mathrm{En}(\mathcal{D})$ is defined as follows.

- $\mathrm{ob}(\mathrm{En}(\mathcal{D})) = \mathcal{V}(1, D_0)$.
- For each $A, B\colon 1 \longrightarrow D_0$, the hom-object $\mathrm{En}(\mathcal{D})(A,B)$ is defined by the pullback

$$\begin{array}{ccc} \mathrm{En}(\mathcal{D})(A,B) & \xrightarrow{\;!\;} & 1 \\ {\scriptstyle\iota_{A,B}}\big\downarrow & \lrcorner & \big\downarrow{\scriptstyle\langle A,B\rangle} \\ D_1 & \xrightarrow{\;\langle\mathrm{dom},\mathrm{cod}\rangle\;} & D_0\times D_0 \end{array}$$

  in $\mathcal{V}$.

- For each $A\colon 1 \longrightarrow D_0$, the morphism j_A is the unique morphism making the diagram

$$\begin{array}{ccccc} 1 & & \xrightarrow{\qquad\mathrm{id}_1\qquad} & & 1 \\ & \overset{j_A}{\searrow} & & & \\ {\scriptstyle A}\big\downarrow & & \mathrm{En}(\mathcal{D})(A,A) & \xrightarrow{\;!\;} & 1 \\ & & \big\downarrow{\scriptstyle\iota_{A,A}} & & \big\downarrow{\scriptstyle\langle A,A\rangle} \\ D_0 & \xrightarrow{\;e\;} & D_1 & \xrightarrow{\;\langle\mathrm{dom},\mathrm{cod}\rangle\;} & D_0\times D_0 \end{array}$$

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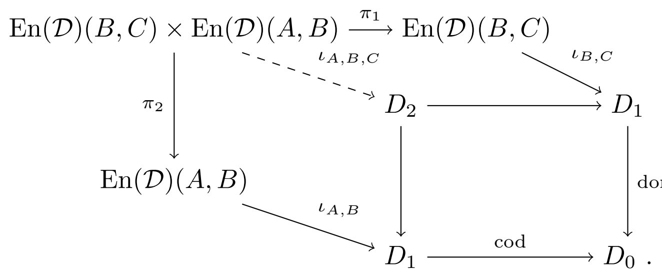
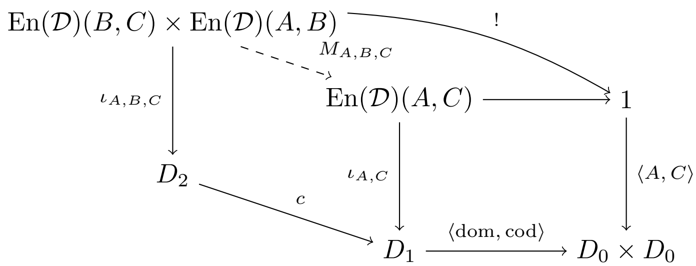
- For each $A, B, C\colon 1 \longrightarrow D_0$, first define $\iota_{A,B,C}$ by the diagram



  Then the morphism $M_{A,B,C}$ is the unique morphism making the diagram



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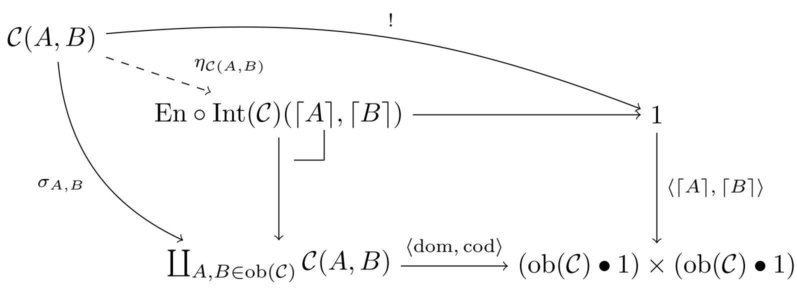
The resulting data defines a $\mathcal{V}$-category. It is routine to check that En extends canonically to a functor $\mathrm{En}\colon \mathbf{Cat}(\mathcal{V}) \longrightarrow \mathcal{V}\text{-}\mathbf{Cat}$.

Proposition 4.2. If $\coprod\colon \mathcal{V}^I \longrightarrow \mathcal{V}/(I \bullet 1)$ preserves binary products for every set I, the functors Int and En form an adjunction $\mathrm{Int} \dashv \mathrm{En}\colon \mathbf{Cat}(\mathcal{V}) \longrightarrow \mathcal{V}\text{-}\mathbf{Cat}$.

Proof. Given a $\mathcal{V}$-category $\mathcal{C}$, the component of the unit of the adjunction $\eta_\mathcal{C}\colon \mathcal{C} \longrightarrow \mathrm{En} \circ \mathrm{Int}(\mathcal{C})$ is defined as follows:

- $\mathrm{ob}(\eta_\mathcal{C})\colon \mathrm{ob}(\mathcal{C}) \longrightarrow \mathcal{V}\text{-}\mathbf{Cat}(1, \mathrm{ob}(\mathcal{C}) \bullet 1)$ is given by $\mathrm{ob}(\eta_\mathcal{C})(C) = \ulcorner C \urcorner$.
- On hom-objects, given A, B in $\mathrm{ob}(\mathcal{C})$, $\eta_{\mathcal{C}(A,B)}$ is the unique morphism making the diagram



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For universality of $\eta_{\mathcal{C}}$ to Int from $\mathcal{C}$, let $\mathcal{D}$ be in $\mathbf{Cat}(\mathcal{V})$ and $f\colon \mathcal{C} \longrightarrow \mathrm{En}(\mathcal{D})$. Define $f'\colon \mathrm{Int}(\mathcal{C}) \longrightarrow \mathcal{D}$ as follows.

- On objects: for each $C \in \mathrm{ob}(\mathcal{C})$, let $f'_C = f(C)\colon 1 \longrightarrow D_0$, and define $f'_0 = [f'_C]_{C\in \mathrm{ob}(\mathcal{C})}\colon \mathrm{ob}(\mathcal{C})\bullet 1 \longrightarrow D_0$.
- On morphisms: for each A, $B \in \mathrm{ob}(\mathcal{C})$, let $f'_{(A,B)} = \iota_{f(A),f(B)} \circ f_{(A,B)}\colon \mathcal{C}(A,B) \longrightarrow D_1$ and define $f'_1 = [f'_{(A,B)}]_{A,B\in\mathrm{ob}(\mathcal{C})}\colon \coprod_{A,B\in\mathrm{ob}(\mathcal{C})} \mathcal{C}(A,B) \longrightarrow D_1$.

That f' is a morphism in $\mathbf{Cat}(\mathcal{V})$ follows from the fact that f is a morphism in $\mathcal{V}$-$\mathbf{Cat}$ and from the definition of $\mathrm{En}(\mathcal{D})$. Furthermore, f' is the unique map such that $\mathrm{En}(f') \circ \eta_{\mathcal{C}} = f$, so $\eta_{\mathcal{C}}$ is universal to Int from $\mathcal{C}$. Hence $\mathrm{Int} \dashv \mathrm{En}$, as required. Q.E.D.

4.3 The category $\mathbf{Cat}_{\mathrm{d}}(\mathcal{V})$: characterising enriched categories

As mentioned in the Introduction, even in the case of $\mathcal{V} = \mathbf{Set} \times \mathbf{Set}$, the functor $\mathrm{Int}\colon \mathcal{V}\text{-}\mathbf{Cat} \longrightarrow \mathbf{Cat}(\mathcal{V})$ is not fully faithful. So, if we are to characterise $\mathcal{V}$-$\mathbf{Cat}$ in terms of categories in $\mathcal{V}$, we need to replace $\mathbf{Cat}(\mathcal{V})$ by a variant. We accordingly define the category $\mathbf{Cat}_{\mathrm{d}}(\mathcal{V})$ of categories in $\mathcal{V}$ with discrete object of objects to be the pullback

$$\begin{array}{ccc} \mathbf{Cat}_{\mathrm{d}}(\mathcal{V}) & \longrightarrow & \mathbf{Cat}(\mathcal{V}) \\ \downarrow & \lrcorner & \downarrow{\scriptstyle (-)_0} \\ \mathbf{Set} & \xrightarrow{\;-\bullet 1\;} & \mathcal{V} \end{array}$$

of categories. So an object of $\mathbf{Cat}_{\mathrm{d}}(\mathcal{V})$ is a pair $(\mathcal{D}, I)$ consisting of a category $\mathcal{D}$ in $\mathcal{V}$ together with a small set I such that D_0, the object of objects of $\mathcal{D}$, is $I \bullet 1$. A morphism in $\mathbf{Cat}_{\mathrm{d}}(\mathcal{V})$ from $(\mathcal{D}, I)$ to $(\mathcal{D}', I')$ consists of a functor in $\mathcal{V}$ from $\mathcal{D}$ to $\mathcal{D}'$ together with a function from I to I' that provides the action of the functor on objects.

One can make evident modifications to Int and En to form functors of the form $\mathrm{Int}'\colon \mathcal{V}\text{-}\mathbf{Cat} \longrightarrow \mathbf{Cat}_{\mathrm{d}}(\mathcal{V})$ and $\mathrm{En}'\colon \mathbf{Cat}_{\mathrm{d}}(\mathcal{V}) \longrightarrow \mathcal{V}\text{-}\mathbf{Cat}$. Note that En' is not simply the restriction of En, since we define $\mathrm{ob}(\mathrm{En}'(\mathcal{D}, I)) = I$. Making the corresponding evident modifications of the proof of Proposition 4.2 allows us to conclude the following.

Proposition 4.3. If $\coprod\colon \mathcal{V}^I \longrightarrow \mathcal{V}/(I \bullet 1)$ preserves finite products for every set I, the functors Int' and En' form an adjunction $\mathrm{Int}' \dashv \mathrm{En}'\colon \mathbf{Cat}_{\mathrm{d}}(\mathcal{V}) \longrightarrow \mathcal{V}\text{-}\mathbf{Cat}$.

Theorem 4.4. Let $\mathcal{V}$ be an extensive cartesian closed category with finite limits. Then $\mathrm{Int}' \dashv \mathrm{En}'\colon \mathbf{Cat}_{\mathrm{d}}(\mathcal{V}) \longrightarrow \mathcal{V}\text{-}\mathbf{Cat}$ is an adjoint equivalence.

Proof. For any $\mathcal{V}$-category $\mathcal{C}$, the assertion that the $\mathcal{C}$-component of the unit of the adjunction is an isomorphism follows from extensivity of $\mathcal{V}$ where $I = \mathrm{ob}(\mathcal{C}) \times \mathrm{ob}(\mathcal{C})$ as in (4), noting that $(\mathrm{ob}(\mathcal{C}) \times \mathrm{ob}(\mathcal{C})) \bullet 1$ is canonically isomorphic to $(\mathrm{ob}(\mathcal{C}) \bullet 1) \times (\mathrm{ob}(\mathcal{C}) \bullet 1)$.

Given an object $(\mathcal{D}, I)$ of $\mathbf{Cat}_{\mathrm{d}}(\mathcal{V})$, the assertion that the $(\mathcal{D}, I)$-component of the counit of the adjunction is an isomorphism follows from extensivity of $\mathcal{V}$ where I is replaced by $I \times I$ in (4), here noting that $(I \times I) \bullet 1$ is canonically isomorphic to $(I \bullet 1) \times (I \bullet 1)$. Q.E.D.

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Theorem 4.4 allows us to describe $\mathcal{V}$-**Cat** in terms of **Cat**$(\mathcal{V})$ as follows.

Corollary 4.5. Let $\mathcal{V}$ be an extensive cartesian closed category with finite limits. The following commutative square of categories is a bipullback in $\mathscr{CAT}$, the 2-category of locally small categories.

$$\begin{array}{ccc} \mathcal{V}\text{-}\mathbf{Cat} & \xrightarrow{\text{Int}} & \mathbf{Cat}(\mathcal{V}) \\ {\scriptstyle \text{ob}(-)}\downarrow & & \downarrow{\scriptstyle (-)_0} \\ \mathbf{Set} & \xrightarrow{-\bullet 1} & \mathcal{V} \end{array}$$

We conclude by extending our leading examples of extensive categories in Section 2 to include cartesian closedness and small limits.

Example 4.1. A presheaf category is an extensive cartesian closed category with small limits.

Example 4.2. Let $\mathcal{V}$ be a cartesian closed category with all small limits. Then the free small coproduct completion **Fam**$(\mathcal{V})$ of $\mathcal{V}$ is an extensive cartesian closed category with small limits.

Example 4.3. Let $\mathcal{V}$ be an extensive cartesian closed category with all small limits. Then the category $\mathcal{V}$-**Cat** of $\mathcal{V}$-categories is again an extensive cartesian closed category with small limits.

Example 4.4. Let $\mathcal{V}$ be an extensive cartesian closed category with all small limits. Then the category **Cat**$(\mathcal{V})$ of categories internal to $\mathcal{V}$ is again an extensive cartesian closed category with small limits. (For cartesian closedness, see [9, Section B.2.3].)

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